Solving the frustrated spherical model with *q*-polynomials

1998 J. Phys. A: Math. Gen. 31 3141

(http://iopscience.iop.org/0305-4470/31/14/004)

J. Phys. A: Math. Gen. **31** (1998) 3141–3151. Printed in the UK PII: S0305-4470(98)88508-9

Solving the frustrated spherical model with q-polynomials

Andrea Cappelli and Filippo Colomo

INFN, Sezione di Firenze and Dipartimento di Fisica, Università di Firenze, Largo E Fermi 2, 50125 Firenze, Italy

Received 17 October 1997

Abstract. We analyse the spherical model with frustration induced by an external gauge field. This has been recently mapped in infinite dimensions onto a problem of q-deformed oscillators, whose real parameter q measures the frustration. We find the analytic solution of this model by suitably representing the q-oscillator algebra with q-Hermite polynomials. We also present a related matrix model which possesses the same diagrammatic expansion in the planar approximation. Its interaction potential is oscillating at infinity with period $\log(q)$, and may lead to interesting metastability phenomena beyond the planar approximation. The spherical model is similarly q-periodic, but does not exhibit such phenomena: actually its low-temperature phase is not glassy and depends smoothly on q.

1. Introduction

Recently Parisi and co-workers [1, 2] introduced and analysed the spherical and XY spin models with frustration, but in the absence of quenched disorder. Their aim was to test the conjecture that the frustrated deterministic systems at low temperature behave like some suitably chosen spin-glass models with quenched disorder [3]. They considered the frustrated models in the limit of large dimensionality D of the lattice, where the saddle-point approximation becomes exact. In their analysis of these models, they showed that the high-temperature expansion can be exactly rewritten by using the q-oscillators algebra [1, 2]. Here q measures the frustration per plaquette and varies continuously between the fully frustrated case ($q = -1$, fermionic algebra) and the ferromagnetic case ($q = 1$, bosonic algebra). Similar q-deformed algebraic relations have also appeared in the Hofstadter problem of quantum particles hopping on a two-dimensional lattice in a magnetic field [4]. This problem is closely related to the frustrated spin models, which can be considered as simplified models of hopping in the large D and classical limits. This relation provides another motivation for our analysis. Finally, we note that the frustrated XY model can also describe Josephson junction arrays in a magnetic field.

In this paper, we solve exactly the frustrated spherical model in the large D limit, both in the high- and low-temperature phases. We use the 'coordinate' representation of the q-oscillators, which is given by the q-Hermite polynomials [5, 6]. We find that the spectrum of the lattice Laplacian is essentially given by a Jacobi theta function; thus, it is periodic along the imaginary axis, with period $\log(q)$. This q-periodicity does not affect the low-temperature phase of the spherical model, which is rather standard and non-glassy, the effect of frustration being quantitative only. On the other hand, we show that the spherical model is associated to a matrix model [7, 8], which has the same diagrammatic expansion in the planar approximation. The potential of this matrix model is oscillating at infinity, where it has infinitely many minima at approximate distance $\log(q)$ (see figure 2). Once the corrections to the planar limit are considered, its states become metastable by tunnelling to these minima: therefore, we argue that this model could have a band spectrum and possibly behave as a spin glass. Similar phenomena could also occur in the frustrated XY model, whose analysis is, however, left to future investigations.

The models we consider are described by the Hamiltonian

$$H_0 = -\frac{1}{\sqrt{2D}} \sum_{\langle jk \rangle} \phi_j^\dagger U_{jk} \phi_k + \text{h.c.} \tag{1}$$

The complex field $\phi_j \in \mathbb{C}$ is defined on the sites (labelled by j) of a D-dimensional hypercubic lattice, and there are nearest-neighbour interactions U_{ij}. Three different models can be obtained by constraining the field as follows

$$\beta H_G = \beta H_0 + \sum_{j=1}^{N} |\phi_j|^2 \tag{2}$$

$$\beta H_S = \beta H_0 + \mu \left(\sum_{j=1}^{N} |\phi_j|^2 - N \right) \tag{3}$$

$$\beta H_{XY} = \beta H_0 + \sum_{j=1}^{N} \mu_j (|\phi_j|^2 - 1). \tag{4}$$

The first is the Gaussian model, which only exists in the high-temperature phase, where the mass term dominates the kinetic term (1). The second is the spherical model [9], which uses the Lagrange multiplier μ to enforce the condition $\sum |\phi_j|^2 = N$, N being the total number of sites. This Lagrange multiplier is promoted to a field μ_j in the XY model (4), whose variables are constrained to live on the unit circle $|\phi_j| = 1$, $\forall j$.

The couplings U_{jk} are complex numbers of modulus one and satisfy the relation $U_{jk} = U_{kj}^*$; they are the link variables of an Abelian lattice gauge field, without kinetic term, which produces a static external magnetic field. This is chosen in such a way as to give the same magnetic flux $\pm B$ for any plaquette of the lattice (the product of the four U's around the plaquette is $\mathrm{e}^{\pm \mathrm{i} B}$). Therefore, the magnetic field should have the same projection on all the axes of the lattice, modulo the sign. In order to avoid the choice of a preferred direction in the lattice, these signs are chosen randomly† [1].

The ferromagnetic spin interaction is obtained for $B = 0$, i.e. $U_{jk} = 1$. Non-vanishing values of B induce a frustration around each plaquette, which is maximal for $B = \pi$, the fully frustrated case. Of course all the intermediate, partially frustrated cases, with $0 < B < \pi$, are also interesting. As explained in [1], the $B = 0$ theory should be treated with care, because the large D limit is different in this case, and the normalization factor $1/\sqrt{2D}$ in (1) should be replaced by $1/(2D)$.

In order to study the previous models, the main difficulty consists of finding the spectrum of the lattice Laplacian in the presence of the magnetic field, which is defined as:

$$(\Delta f)_j = \sum_{k=1}^{2D} U_{jk} f_k. \tag{5}$$

Similarly to the Hofstadter problem [4], there is a competition between the periodicity due to the lattice and those induced by commensurate magnetic fields of the form $B = 2\pi r/s$, with

† This is a small amount of randomness, of order $D(D-1)/2$, to be compared with the $\sim L^D$ randomness present in systems with quenched disorder.

r, s integers. Thus, we could expect a complex band structure in the spectrum. However, there are simplifications due to the large D limit. The authors of [1, 2] approached the problem via the high-temperature expansion. In the case of the free energy of the Gaussian model (equation (2)), this is given by a sum over all closed loops, as follows

$$\beta F = \sum_{n=0,\text{even}}^{\infty} \frac{1}{n} \left(\frac{\beta}{\sqrt{2D}} \right)^n \mathcal{N}(n) \langle W(\mathcal{C}) \rangle_n = \sum_{k=0}^{\infty} \frac{\beta^{2k}}{2k} G_k. \tag{6}$$

In this expression, the loops $\mathcal{C}$ are arranged according to their length $n = 2k$, and their number is $\mathcal{N}(n)$. For each loop, the magnetic field yields a weight, which is given by the Wilson loop $W(\mathcal{C})$, the path-ordered product of the couplings U along the loop. The brackets $\langle\ \rangle_n$ represent the average of this weight over all the $\mathcal{N}(n)$ circuits of length n. In the second expression of equation (6), we introduce the notation G_k for the product of the multiplicity and the Wilson loop average.

Each loop encloses a number of plaquettes and receives a weight proportional to $\exp(\mathrm{i}BA)$, where A is the sum of plaquettes with signs depending on the orientations [10]. Due to the average over orientations and loops, the quantity G_k is a polynomial in the variable

$$q = \cos B \tag{7}$$

of order $k(k-1)/2$, which is given by the maximal area enclosed by the loop. Parisi [1] enumerated these diagrams in the large D limit. First he showed that the same counting is given by the Feynman diagrams with $2k$ external points, which are joined pairwise by lines (propagators) intersecting I times, and have assigned the weight q^I. These diagrams also occur in the topological (large N) expansion of matrix models [7], where the planar limit corresponds to no intersections, i.e. to the $q = 0$ case. Secondly, Parisi found a recursion relation for the coefficients of the polynomial $G_k(q)$—a sort of Wick theorem—which can be accurately expressed by the algebra of the q-oscillators $a_q, a_q^\dagger$:

$$a_q a_q^\dagger - q a_q^\dagger a_q = 1. \tag{8}$$

These operators [11] act on the Hilbert space spanned by the vectors: $|m\rangle$, $m = 0, 1, \ldots,$ as follows

$$\begin{aligned} a_q^\dagger |m\rangle &= \sqrt{[m+1]_q}\,|m+1\rangle \\ a_q |m\rangle &= \sqrt{[m]_q}\qquad |m-1\rangle \qquad a_q|0\rangle = 0 \end{aligned} \tag{9}$$

where

$$[m]_q = \frac{1-q^m}{1-q}. \tag{10}$$

Using the recursion relation, the weighted multiplicities of the diagrams of equation (6) were neatly written as an expectation value over the ground state of the q-oscillators [1, 2]:

$$G_k(q) = \langle 0 | (a_q^\dagger + a_q)^{2k} | 0 \rangle. \tag{11}$$

2. Spectrum of the Laplacian and high-temperature expansion

We shall now make expression (11) more explicit by introducing the coordinate representation for the q-oscillators. First note that in the Gaussian model (2) the quantity G_k is nothing else than the trace of $(2k)$th power of the frustrated Laplacian, such that we can write the general relation

$$\mathrm{Tr}[f(\Delta)] \equiv \langle 0 | f(x_q) | 0 \rangle \qquad x_q \equiv a_q^\dagger + a_q. \tag{12}$$

The x_q coordinate representation, $x_q|x\rangle = x|x\rangle$, is given by the so-called continuous q-Hermite polynomials [6, 12]. These are defined by:

$$H_n(x) = \langle x|n\rangle \mathcal{C}_n \qquad \mathcal{C}_n = ([n]_q!)^{1/2}\mathcal{C}_0 \tag{13}$$

where the normalization constant $\mathcal{C}_0$ is fixed by $H_0(x) = 1$ and the q-factorial is

$$[n]_q! = [n]_q[n-1]_q \ldots [1]_q \qquad [1]_q = [0]_q = 1. \tag{14}$$

These polynomials satisfy, of course, a three-term recursion relation in the index n:

$$xH_n(x) = H_{n+1}(x) + [n]_q H_{n-1}(x) \qquad n \geqslant 1. \tag{15}$$

Moreover, they obey a q-difference equation [4]† in their coordinate x, which ranges over the interval $x \in [-2/\sqrt{1-q}, 2/\sqrt{1-q}]$. A convenient parametrization is

$$x = \frac{2}{\sqrt{1-q}}\cos\theta \qquad \theta \in [0, \pi]. \tag{16}$$

More properties of these q-Hermite polynomials can be found in [6], where they are defined as $\mathcal{H}_n(\cos\theta) = (1-q)^{n/2}H_n(x)$. The most important property for us is the orthogonalizing measure $\nu_q(x)$ [5, 6, 12]:

$$\int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}} \nu_q(x)\,\mathrm{d}x\, H_n(x)H_m(x) = \delta_{n,m}[n]_q! \tag{17}$$

$$\begin{aligned}
\nu_q(x) &= \frac{\sqrt{1-q}}{2\pi} q^{-1/8}\Theta_1\left(\frac{\theta}{\pi}, q\right) \\
&= \frac{\sqrt{1-q}}{\pi}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}\sin[(2n+1)\theta] \\
&= \frac{\sqrt{1-q}}{\pi}\sin\theta\prod_{n=1}^{\infty}(1-q^n)(1-q^n\mathrm{e}^{2\mathrm{i}\theta})(1-q^n\mathrm{e}^{-2\mathrm{i}\theta})
\end{aligned} \tag{18}$$

where $\Theta_1(z, q)$ is the first Jacobi theta function.

Using these results, we can rewrite the general ground-state expectation value in equation (12) as follows

$$\mathrm{Tr}[f(\Delta)] \equiv \int \mathrm{d}x\,\nu_q(x) f(x). \tag{19}$$

Therefore, we have shown that the Laplacian is diagonal in the coordinate representation of the q-oscillators: it has a continuous spectrum over the interval $[-2/\sqrt{1-q}, 2/\sqrt{1-q}]$, with eigenvalue density given by ν_q in equation (18).

Let us discuss this spectrum in some interesting limits. For $q = 0$, we find $\nu_0(x) = 1/\pi\sqrt{1-x^2/4}$, which is the Wigner semicircle law for the Gaussian Hermitian matrix model [7]. This results confirms the previous correspondence between the high-temperature expansion of the $q = 0$ Gaussian model and the planar Feynman diagrams without interaction vertices of the matrix model. Actually, this correspondence can be extended to all values of q (see section 4). Note also that the $q = 0$ frustrated model is diagrammatically equivalent to the gauge spin glass where the coupling U_{jk} are random quenched variables [1]. Next, for $q = -1$, the measure becomes a representation for $(\delta(x-1) + \delta(x+1))/2$, and we recover the two states of the fermionic algebra [5]. The limit $q \to 1$ is singular, owing to the $D \to \infty$ peculiarities stated before; nevertheless, $\nu_1(x)$ becomes the Gaussian distribution, after multiplicative renormalization [5].

† This q-periodicity will be discussed better later.

Let us now check that our expression for the Laplacian reproduces the high-temperature expansion computed in [1, 2]. The free energy of the Gaussian model (2) can be written in terms of the Laplacian as $\beta F = -\log Z = \mathrm{Tr}[\ln(1-\beta\Delta)]$. Therefore, the internal energy $U(\beta)$ is:

$$1 - \beta U(\beta) = R(\beta) \qquad R(z) \equiv \mathrm{Tr}\left[\frac{1}{1-z\Delta}\right]. \tag{20}$$

In the last equation, we introduced the resolvent $R(z)$, which will play a major role in the subsequent study of the spherical model. $R(z)$ is originally well defined for real z in the interval $|z| < z_c = (\sqrt{1-q})/2$ and is then analytically extended to the whole complex plane of z. By using the results of the previous section, we find:

$$\begin{aligned} R(z) &= \int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}} \mathrm{d}x \, \frac{\nu_q(x)}{1-zx} \\ &= \frac{\sqrt{1-q}}{z} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \left[K\left(\frac{2z}{\sqrt{1-q}}\right)\right]^{2n+1} \end{aligned} \tag{21}$$

where we introduced

$$K(\alpha) = \frac{1-\sqrt{1-\alpha^2}}{\alpha}. \tag{22}$$

The singularities of $R(z)$ in the complex plane are completely determined by those of the simpler function $K(2z/\sqrt{1-q})$. Actually, the series is very well convergent for $|q| < 1$ as any Jacobi theta function. Moreover, for $|q| = 1$, it is a geometric series, which is still convergent because $|K| < 1$ for $|z| < z_c$.

The high-temperature expansion of the internal energy of the Gaussian model can be obtained by expanding $R(\beta)$ in series; there only appear even powers of β with q-dependent coefficients. We have computed the series to $\mathrm{O}(\beta^{18})$ with Mathematica [14], and obtained the polynomials $G_k(q)$, $k = 1, \ldots, 9$ of equation (6). We have verified that they indeed match the results of [1, 2], which were found by direct enumeration of the graphs on a computer.

For $q = 0$, we have $U(\beta) = (\beta - K(2\beta))/\beta^2$ and the high-temperature expansion is singular at $\beta_c = \frac{1}{2}$, which is the critical temperature of the Gaussian model. The specific heat diverges as $(\beta_c - \beta)^{-1/2}$ at the transition, i.e. the value of critical exponent α is $\frac{1}{2}$. This result still holds for generic $-1 < q < 1$ as a consequence of the relation between the singularities of $R(z)$ and $K(z)$.

3. Solution of the spherical model

We are now ready to solve the spherical model. Our approach essentially follows the original Berlin–Kac solution of the ferromagnetic model in $D = 1, 2, 3$ [9]. Indeed we shall see that the mechanism of the phase transition is the same, only the form of the Laplacian is different. The partition function is

$$\begin{aligned} Z &= \int_{\mu_0 - \mathrm{i}\infty}^{\mu_0+\mathrm{i}\infty} \mathrm{d}\mu \int \mathcal{D}\phi \exp\left[\beta \sum \phi^\dagger \Delta \phi - \mu \left(\sum |\phi|^2 - N\right)\right] \\ &= \int_{\mu_0 - \mathrm{i}\infty}^{\mu_0+\mathrm{i}\infty} \mathrm{d}\mu \, \exp[-\mathrm{Tr}[\ln(\mu - \beta\Delta)] + \mu N]. \end{aligned} \tag{23}$$

The integration path over μ is along a straight line which runs in the complex plane parallel to the imaginary axis; a positive real part has been added in order to make the integration over ϕ convergent. The trace in the exponent is proportional to N (Δ is an N-dimensional operator): in the large N limit, we can apply the saddle-point method to evaluate the integral over μ. The saddle-point equation is:

$$\mathrm{Tr}\left[\frac{1}{\mu - \beta\Delta}\right] = 1 \tag{24}$$

and must be solved for real, positive values of μ. Introducing the variable $z = \beta/\mu$, this equation can be written in terms of the resolvent (20), as follows

$$\frac{\beta}{z} = R(z) \qquad z \equiv \frac{\beta}{\mu}. \tag{25}$$

In the case of $q = 0$, this equation can be easily solved:

$$\frac{1}{2z^2}(1 - \sqrt{1-4z^2}) = \frac{\beta}{z} \longrightarrow z(\beta) = \frac{\beta}{1+\beta^2} \tag{26}$$

which corresponds to $\mu = 1 + \beta^2$. This solution is valid for $z < z_c = \frac{1}{2}$, namely it describes the high-temperature phase $\beta < \beta_c = 1$. Indeed, let us follow the saddle point in the z plane, starting from $\beta = 0$, where $z = 0$ as well. Upon increasing β, z moves towards the critical value $z_c = \frac{1}{2}$; for $\beta > 1$, z can no longer increase because it finds the square-root branch cut of $R(z)$, and therefore 'it sticks to the singularity' [9]. Hence, for $\beta > 1$, the saddle-point equation no longer has an acceptable solution; nevertheless, the leading contribution to the μ integral in (23) still comes from the neighbourhood of $z = \frac{1}{2}$. Therefore, the low-temperature solution is $z(\beta) = \frac{1}{2}$ independent of β, i.e. $\mu = 2\beta$. The free and internal energies are therefore given by

$$\beta F = \mathrm{Tr}[\ln(\mu - \beta\Delta)] - \mu \tag{27}$$

$$\begin{aligned} U(\beta) &= (\partial_\beta + (\partial_\beta \mu)\partial_\mu)(\beta F) \\ &= \frac{1}{\beta}\left(1 - \mathrm{Tr}\left[\frac{1}{1 - z\Delta}\right]\right) + \left(\frac{1}{\mu}\,\mathrm{Tr}\left[\frac{1}{1-z\Delta}\right] - 1\right)\frac{\partial\mu}{\partial\beta}. \end{aligned} \tag{28}$$

The last term in this equation vanishes in the high-temperature phase, because the saddle-point condition is satisfied; however, it should be included for the low-temperature regime. As a matter of fact, the result can be written in both phases as follows

$$U(\beta) = \frac{1}{\beta} - \frac{1}{z(\beta)}. \tag{29}$$

More explicitly,

$$U(T) = \begin{cases} -\dfrac{1}{T} & T > 1 \\ T - 2 & T < 1 \end{cases} \qquad (q = 0). \tag{30}$$

The internal energy is therefore continuous at the transition, together with its first derivative. However, the specific heat presents a discontinuity in its first derivative. These results are qualitatively very similar to those of the Berlin–Kac analysis [9]; nevertheless, their model is quite different, in the sense that frustration is completely absent there.

The solution for any $-1 < q < 1$ is a straightforward generalization of the $q = 0$ case. The discussion of the saddle-point equation (25) is analogous, because the singularities of $R(z)$ are still given by a square-root branch cut. Although we cannot find an explicit expression for $z(\beta)$, we can follow its behaviour: starting from low β, where $z \sim \beta$, the saddle point moves towards higher values of z until it hits the cut of $R(z)$ at

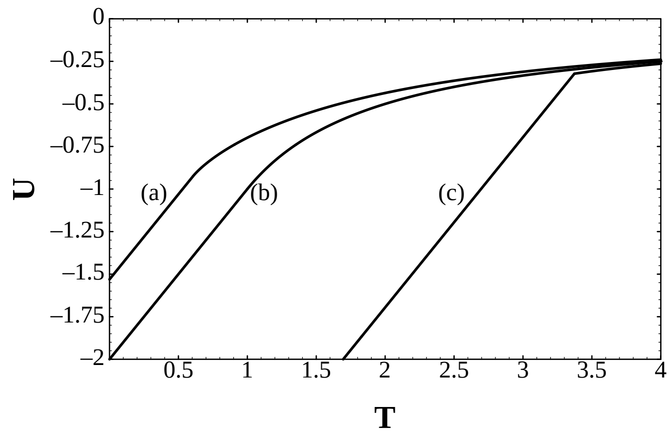

Figure 1. Internal energy of the spherical model versus temperature for three values: (a) $q = -0.707$; (b) $q = 0$; (c) $q = 0.707$.

$z = z_c = (\sqrt{1-q})/2$. In the low-temperature phase, the value of z sticks to z_c and thus $\mu = \beta/z_c$ for $\beta > \beta_c$. The critical temperature is given by

$$\beta_c = z_c R(z_c) = \sqrt{1-q} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}. \tag{31}$$

The internal energy is still given by the general formula (29), which now reads

$$U(T) = \begin{cases} T - \dfrac{2}{\sqrt{1-q}} & T < T_c \\ -\dfrac{1}{T} - q\dfrac{1}{T^3} - (q^3 + 3q^2)\dfrac{1}{T^5} + \mathrm{O}\left(\dfrac{1}{T^7}\right) & T \gg T_c \end{cases} \qquad (-1 < q < 1). \tag{32}$$

A plot of the internal energy is shown in figure 1 for the three values $q = (-1/\sqrt{2}, 0, 1/\sqrt{2})$. This can easily be obtained without solving the saddle-point equation, by using $z \in [0, z_c]$ as a parameter for both $U(\beta)$ in equation (29) and β in equation (25). Note that for $q \neq 0$, the first derivative of the internal energy is not continuous at the transition. The critical temperature goes to infinity for $q \to 1$ and to zero for $q \to -1$, respectively. In these limiting cases, we can sum the series in $R(z)$ (21) and obtain the explicit form of $z(\beta)$ and the internal energy. For $q = 1$, we find the meaningless results $z = \beta$ and $U(T) = 0$ in the high-temperature phase, which collapses to a point ($T_c = \infty$): this case has a pathologic $D \to \infty$ limit, as discussed in [1], and a sensible theory should include $1/D$ corrections.

The limit $q = -1$ is well defined and corresponds to the fully frustrated model at $D = \infty$†. Here, the high-temperature phase extends down to $T = T_c = 0$, i.e. there is no transition. Actually, we find that the resummed $R(z)$ no longer has the square-root branch cut of $K(z)$ and that the mapping $\beta = \beta(z)$ is invertible on the whole positive β axis:

$$\begin{aligned} \beta &= \frac{z}{1-z^2} \qquad (q = -1) \\ U &= -z = \frac{1}{2\beta}\left(1 - \sqrt{1+4\beta^2}\right). \end{aligned} \tag{33}$$

† Note that the limits $D \to \infty$ and $q \to -1$ do not commute, as discussed in [1].

Let us remark that the absence of a phase transition has also been observed in the fully frustrated long-range XY model [13].

In conclusion, the qualitative behaviour of the (infinite-dimensional) frustrated spherical model in the low-temperature phase is rather smooth and standard: indeed, this model does not present the potential features of: (i) glassy behaviour (many ground states), which is usually found in systems with quenched disorder; (ii) commensurability in the spectrum for $B = 2\pi r/s$, which was observed in the $D = 2$ hopping model in a magnetic field. In view of point (i), we would like to remark that the corresponding model with quenched disorder, i.e. the spherical gauge glass, does not exhibit a glassy phase either. Actually, this model corresponds diagrammatically to the $q = 0$ frustrated model. In intuitive terms, the spin variables are only loosely constrained by the spherical condition, $\sum_i |\phi_i|^2 = N$, and thus can globally adapt themselves to any complex coupling configuration. In view of (ii), the commensurability effects are probably washed out by the $D \to \infty$ limit, which oversimplifies the geometry of hopping on the lattice.

On the other hand, we expect a glassy phase in the frustrated XY model [2]. In this model, the Lagrange multiplier μ becomes a local field μ_j and the saddle-point equation (24) is functional. Nothing changes in the high-temperature phase, where the solution $\mu_j = \mu =$ constant is correct. However, the solution(s) in the low-temperature phase can be rather different from the one of the spherical model, and will not be discussed here. Rather, we shall approach this problem from a different perspective, by establishing a relation with the well-developed subject of matrix models.

4. Analogy with the matrix models

Let us first recall the solution of the Hermitian matrix models in the the large N approximation, which corresponds to the planar Feynman diagrams [7]. These models are characterized by the $(N \times N)$ matrix variable $M = M^\dagger$ and by the Hamiltonian $\beta H = \mathrm{Tr}\, V(M) = \mathrm{Tr}(M^2/2 + \cdots)$. After diagonalization of the matrix, one is left with the partition function over the eigenvalues λ_i, $i = 1, \ldots, N$:

$$Z_M = \int \prod_{i=1}^{N} \mathrm{d}\lambda_i \, \mathrm{e}^{-\sum_i V(\lambda_i)} \prod_{i<j} (\lambda_i - \lambda_j)^2. \qquad (34)$$

This can be thought of as being the statistical mechanics of N charges with coordinates λ_i in one dimension, which repel each other logarithmically and are kept together by the external potential $V(\lambda)$. This one-dimensional gas of charges usually has a unique phase. In the large N limit, the partition sum is dominated by the contribution of the saddle point, which corresponds to the equilibrium configuration of the charges, neglecting fluctuations. Moreover, the charges become a continuum with density $\nu(\lambda)$, which is normalized to one by a convenient rescaling [7]. The saddle-point equation is:

$$\frac{1}{2} V'(\lambda) = P \int_{-a}^{a} \mathrm{d}x \, \frac{\nu(x)}{\lambda - x} \qquad \lambda \in (-a, a) \qquad \int_{-a}^{a} \mathrm{d}x \, \nu(x) = 1 \qquad (35)$$

where P stands for the principal value of the integral. This is an equation for the unknown $\nu(x)$, as a function of the given potential $V(\lambda)$. Following [7], we introduce the function

$$F(\lambda) = \int_{-a}^{a} \mathrm{d}x \, \frac{\nu(x)}{\lambda - x} \qquad (36)$$

with λ taking values in the complex plane. This function is analytic outside the segment of the real axis $(-a, a)$ corresponding to the spectrum; moreover, it goes to zero at infinity as

$1/\lambda$, due to the normalization of ν. For λ inside the spectrum, we have

$$\mathrm{Re}\, F(\lambda) = \tfrac{1}{2}V'(\lambda) \qquad \mathrm{Im}\, F(\lambda) = -\pi\nu(\lambda) \qquad \lambda \in (-a, a). \tag{37}$$

The saddle-point equation is thus equivalent to these relations for the function $F(\lambda)$: they can usually be solved by analyticity arguments, and determine the density $\nu(\lambda)$.

Here we would like to remark that these formulae are rather similar to those encountered in the spherical model. Actually, we can identify the two saddle-point equations (25) and (35) as follows

$$F(\lambda) \equiv \frac{1}{\lambda} R\left(\frac{1}{\lambda}\right) \qquad z \equiv \frac{1}{\lambda}. \tag{38}$$

More precisely, the saddle-point equation for the spherical model is discussed for λ outside the spectrum, $\nu(\lambda)$ is given and $\beta = \beta(z)$ is the unknown. On the other hand, in the matrix model λ is inside the spectrum, $V(\lambda)$ is given and $\nu(\lambda)$ is the unknown.

We can now define the matrix model corresponding to the frustrated spherical model, as the one which possesses the same eigenvalue density $\nu_q(\lambda)$, equation (18), in the planar limit. By using equations (36) and (37), we can determine its potential:

$$V'(\lambda) = 2\,\mathrm{Re}\, F(\lambda) = 2\sqrt{1-q}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}\cosh[(2n+1)\chi] \tag{39}$$

where we introduced the convenient parametrization

$$\lambda = \frac{2}{\sqrt{1-q}}\cosh\chi \tag{40}$$

for λ outside the cut $(-2/\sqrt{1-q}, 2/\sqrt{1-q})$. The Gaussian matrix model with $V(\lambda) = \lambda^2/2$ is indeed recovered for $q = 0$.

Relation (38) between the spherical and matrix models can also be understood at the level of diagrammatic expansions. In the former model, $R(z)$ generates the high-temperature expansion, whose diagrams were cast into the form of n lines joining $2n$ points with weight q for each intersections [1]. In the latter model, $F(1/z)/z$ is the generating function for the observables $\langle \mathrm{Tr}\, M^{2n}\rangle = \langle \sum_i \lambda_i^{2n}\rangle$ [7]. Their diagrams also have $2n$ external points: in the Gaussian model ($q = 0$), they are joined by propagator lines with no intersections, due to the planar limit. For $q \neq 0$, the spherical model diagrams have non-planar intersections, which are reproduced by interaction vertices in the planar matrix model diagrams; actually, the potential $V(\lambda)$ contains interactions of any order.

The spherical model in the high-temperature phase and the matrix model in its planar limit have corresponding saddle-point equations but rather different free energies, so they are not quite the same physical problem. The analogy with matrix models is, nevertheless, interesting because it could provide some useful technology [8] for solving the XY model. Furthermore, the matrix model itself could exhibit interesting physics beyond the planar approximation: in the following, we shall put forward some educated guesses which are based on the well known physical picture of the gas of charges.

The properties of the matrix model potential $V(\lambda)$ can be found by analysing equation (39). A plot of $V'(\chi)$ is shown in figure 2, for $q = -1/\sqrt{2}$: away from the origin, an oscillating behaviour sets in with period† $\chi \to \chi - (\log q)$, and amplitude growing to infinity. This *q-periodicity* of the potential can be found by inspection of (39), and reads:

$$V'(q^{-1/2}\mathrm{e}^{\chi}) = 2\left(\frac{1-q}{q}\right)^{1/2}\mathrm{e}^{\chi} - q^{-1/2}\mathrm{e}^{2\chi}V'(\mathrm{e}^{\chi}). \tag{41}$$

† The period is $|\log q|/2$ for $0 < q < 1$.

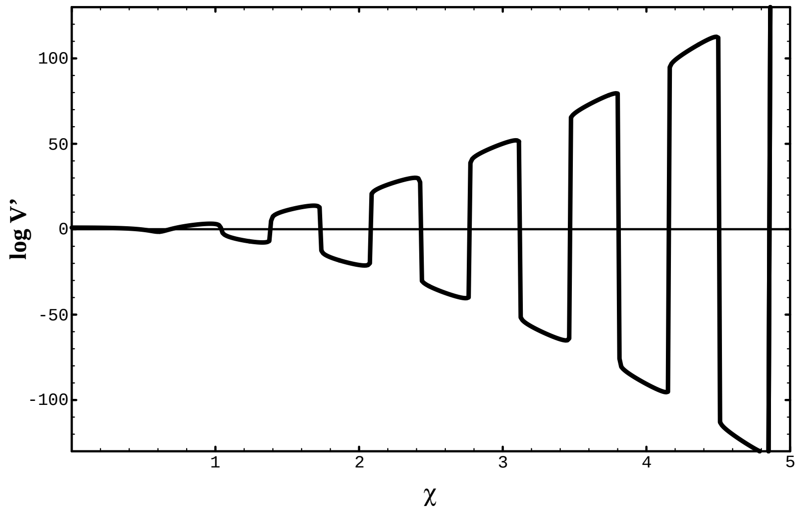

Figure 2. The first derivative of the matrix model potential in equation (39) as a function of χ, for $q = -0.707$.

This is not an exact periodicity due to the presence of the additive term. However, this term becomes negligible for χ far away from the origin, i.e. $\lambda \gg 1$, because V' grows more than exponentially (the amplitude of fluctuations is of $\mathrm{O}(\exp(2\chi^2/|\log q|))$). The q-periodicity of the potential corresponds to a true periodicity of the eigenvalue density $\nu_q(\theta) \propto \Theta_1(\theta/\pi, q)$, because these two quantities are the real and imaginary parts of the function F in (36), respectively (their variables (x, θ) in (16) and (λ, χ) in (40) are also related by analytic continuation). Actually, the eigenvalue density is periodic in the direction of the imaginary θ-axis: $\nu_q(\theta - (\mathrm{i}/2)\log q) \sim \nu_q(\theta)$, for $0 < q < 1$, and $\nu_q(\theta + \pi/2 - (\mathrm{i}/2)\log q) \sim \nu_q(\theta)$, for $-1 < q < 0$.

This periodicity can be interpreted as the potentiality for metastable states, which, however, are not realized in the spherical model, owing to its simplified dynamics. Indeed, its saddle-point equation involves the function $F(\lambda)$, which is also q-periodic, as its real part V'; however, $F(\lambda)$ never grows sufficiently high to develop the oscillating behaviour and, in fact, goes monotonically to zero at infinity.

On the other hand, the states might become metastable in the matrix model beyond the planar approximation. In this approximation, the charges form an equilibrium configuration determined by the minimum of $V(\lambda)$ at the origin, and the tunnelling of charges to lower nearby minima is suppressed [8]: thus, the q-periodicity of the potential is not felt. Beyond this approximation, tunnelling switches on and the states in the spectrum can become metastable. Clearly, a detailed analysis is necessary to understand the effect of tunnelling into a q-periodic set of local minima separated by ever-rising barriers. This might lead to a complex pattern of metastability, which is a characteristic of the glass phase.

In conclusion, there is the possibility that this matrix model might describe some of the expected effects of frustrated magnetic systems in finite dimension D [3, 4]. Moreover, it should be solvable by known techniques [8]. The frustrated XY model is another system which might develop these effects in the low-temperature phase; it would be interesting to pursue the relation between the XY model and the matrix model beyond the planar limit.

Acknowledgments

We would like to thank Philippe Di Francesco, Enzo Marinari, Giorgio Parisi and Paul Wiegmann for useful discussions. This work was supported in part by the European Community Programme 'Training and Mobility of Researchers' FMRX-CT96-0012.

References

[1] Parisi G 1994 *J. Phys. A: Math. Gen.* **27** 7555
[2] Marinari E, Parisi G and Ritort F 1995 *J. Phys. A: Math. Gen.* **28** 4481
[3] Mezard M, Parisi G and Virasoro M A 1987 *Spin Glass Theory and Beyond* (Singapore: World Scientific)
Parisi G 1992 *Field Theory, Disorder and Simulations* (Singapore: World Scientific)
[4] Wiegmann P B and Zabrodin A V 1994 *Nucl. Phys.* B **422** 495
[5] van Leeuwen H and Maassen H 1995 *J. Math. Phys.* **36** 4743
Bozejko M, Kümmerer B and Speicher R 1997 *Commun. Math. Phys.* **185** 129
[6] Gasper G and Rahman M 1990 *Basic Hypergeometric Functions* (Cambridge: Cambridge University Press)
[7] Brezin E, Itzykson C, Parisi G and Zuber J B 1978 *Commun. Math. Phys.* **59** 35
[8] Di Francesco P, Ginsparg P and Zinn-Justin J 1995 *Phys. Rep.* C **254** 1
[9] Berlin T H and Kac M 1952 *Phys. Rev.* **86** 821
Baxter R 1982 *Exactly Solvable Models in Statistical Mechanics* (New York: Academic)
[10] Ademollo M, Cappelli A and Ciafaloni M 1985 *Nucl. Phys.* B **259** 429
[11] Biedenharn L 1989 *J. Phys. A: Math. Gen.* **22** 4581
Macfarlane A 1989 *J. Phys. A: Math. Gen.* **22** L873
Chaichian M and Kulish P 1990 *Phys. Lett.* B **234** 72
[12] Szëgo G 1926 *Sitz. Preuss. Akad. Wiss. Phys. Math.* **19** 242
[13] Marinari E, Parisi G and Ritort F 1995 *J. Phys. A: Math. Gen.* **28** 327
[14] Wolfram S 1991 *Mathematica* (New York: Addison-Wesley)